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Chronicle Development of Fuzzy Metric Spaces

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Abstract : The purpose of this paper is to study the chronicle development of fuzzy sets, fuzzy metric space, intuitionistic fuzzy metric space, fixed points in fuzzy metric Space and common fixed points in fuzzy metric Spaces. In this paper we provide account of some of the fundamental features of fuzzy metric spaces and its various generalizations. In this paper we analyze some results and questions obtained in recent years in fuzzy metric spaces.

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Key words: Fuzzy metric space, intuitionistic fuzzy metric space, t-norm, t-conorm, fuzzy sets.

I. INTRODUCTION

Conventionally a statement is either true or false, with nothing in between. But in real life situations are not deterministic and so cannot be described precisely. Practically the information we collect from real life situations are in general unclear and uncertain so every formal description of the real world is only approximation or idealization of the actual state. Many decision and problem solving tasks are too complex to be understood quantitatively.

In 1965 Iranian Mathematician Prof. L. A. Zadeh[26] first introduced the concept of fuzzy set. The notions like fuzzy set, fuzzy language etc. enable us to handle the degree of uncertainty in a purely mathematical way.

The theory of fuzzy logic is basically a theory of graded concept. In fuzzy set everything has a degree of membership. Fuzzy logic is able to express the amount of ambiguity in human thinking and subjectivity in a comparatively accurate manner. It is specifically designed to mathematically represent uncertainty and vagueness and provide formalized tools for dealing with imprecision of many problems.

Since knowledge can be expressed in a more natural way by fuzzy sets, many engineering and mathematical problems can be easily simplified using Fuzzy set. Fuzzy set theory and fuzzy fixed point theory has numerous applications in applied sciences and engineering such as neural network theory, stability theory, mathematical programming, modeling theory, medical sciences, image processing, control theory, computer science, artificial intelligence, operation research etc.

In Mathematics a subset A of X can be equivalently represented by its Characteristic function, a mapping X_A from the universe X, containing A to the 2 - value set {0,1}. That is to say x belongs to A if and only if $X_A(x) = 1$. But the concept of fuzzy logic used the unit interval [0, 1] instead of {0, 1}. That is in "fuzzy" case the "belonging to" relation $X_A(x)$ between x and A is not "either 0 or 1" but it has a membership degree, between [0, 1].

There are many situations where the distances between the points are rather inexact than being a single non negative real number which led to the introduction of probabilistic metric space. When the uncertainty is due to fuzziness rather than randomness, then the concept of fuzzy metric space is relatively more suitable.

II. DEVELOPMENT

Probabilistic metric spaces were introduced by K. Menger [13] who generalized the theory of metric spaces. In the Menger's theory the concept of distance is considered to be statistical or probabilistic, i.e. he proposed to associate a distribution function F_{xy}, with every pair of elements x, y instead of associating a number, and for any positive number t, interpreted $F_{xy}(t)$ as the probability that the distance from x to y be less than t.

He stated that a probabilistic metric space (PM space) is a pair (X, F) such that X is a nonempty set and F is a mapping from $X \times X$ into Δ^+, whose value F(x, y) denoted by F_{xy}, satisfies for all $x, y, z \in X$:

(PM1) $F_{xy}(t) = 1$ for all $t > 0$ if and only if $x = y$.

(PM2) $F_{xy} = F_{yx}$

(PM3) If $F_{xy}(t) = 1$ and $F_{yz}(s) = 1$, then

$F_{xz}(t + s) = 1$.

In the year 1965 Professor Lofti Zadah[26] was the first person who introduced the concept of fuzzy set. Fuzzy set is characterized by a membership function which assigns each object to a grade of membership between zero and one. He established the relations of inclusion, exclusion, union, intersection, complement, relation for fuzzy set and proved theorem of separation for convex fuzzy sets. Mathematically fuzzy set M on arbitrary set

X is a function (Degree of membership) from X to the unit interval [0, 1] i.e. M: X → [0, 1].

The concept of fuzziness found place in probabilistic metric spaces. The main reason behind this was that, in some cases, uncertainty in the distance between two points was due to fuzziness rather than randomness.

Schweizer and Sklar[21] defined the continuous t-norm.

t-norm: A binary operation $*$: $[0,1]\times[0,1]\to[0,1]$ is called continuous t-norm if it satisfies the following conditions:

1) $*$ is commutative and associative.

2) $*$ is continuous.

3) $a * 1 = 1$. $\forall a \in [0,1]$

4) $a * b \le c * d$ Whenever $a \le c$ and $b \le d$. $\forall a,b,c,d \in [0,1]$.

A Menger space is a triple (X, F, $*$) such that (X,F) is a probabilistic metric space and $*$ is a t-norm such that for all x, y, z ∈ X and t , s ≥ 0:

$F_{xz}(t + s) \ge F_{xy}(t) * F_{yz}(s)$

Heilpern [10] introduced the concept of a fuzzy mapping and proved a fixed point theorem for fuzzy contraction mappings which is a generalization of the fixed point theorem for multi valued mappings of Nadler [16].

In 1975 Kramosil and Michalek [12] applied the concept of fuzziness to the classical metric space and compare the fuzzy metric space with probabilistic metric space, the generalization of metric space.

Mathematically, A fuzzy metric space is an ordered triple (X, M, $*$) such that X is a non empty set, M is a fuzzy set on $X\times X\times[0, \infty)$ satisfying the following conditions, for all x, y, z ∈ X and s, t > 0:

1) M(x, y, 0)= 0;

2) M(x, y, t)=1 if and only if x=y;

3) M(x, y, t)= M(y, x, t);

4) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$;

5) M(x, y, .): $[0, \infty)\to[0, 1]$ is left continuous.

From the above axioms one can show that M_{xy} is an increasing function.

Deng[8], Fang[9], Kaleva and Seikkala[11] also defined Fuzzy metric space in different ways using the concept of fuzzy sets.

By modifying the separation condition and strengthening some conditions of Kramosil and Michalek[12], in 1994, George and Veeramani [3] have defined a special class of fuzzy metric space. They extended the concept of fuzzy metric space with the help of continuous t-norm and introduced Housdorff topology in a fuzzy metric space. They proved that every closed ball is a closed set in fuzzy metric space. They also proved Baire's theorem for fuzzy metric space.

A fuzzy metric space in sense of GV is an ordered triple (X, M, $*$) such that X is a non empty set, M is a fuzzy set on $X\times X\times[0, \infty)$ satisfying the following conditions, for all x, y, z ∈ X and s,t > 0:

1) M(x, y, t)>0;

2) M(x, y, t)=1 if and only if x=y;

3) M(x, y, t)= M(y, x, t);

4) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$;

5) M(x, y, .):$[0, \infty)\to[0, 1]$ is continuous.

M(x, y, t) is considered as the degree of nearness of x and y with respect to t .

They proved that every metric d of metric space (X, d) induces a fuzzy metric under the relation,

$$M(x, y, t)=\frac{t}{t+d(x,y)} \quad \text{for all } t>0 .$$

The real function M_{xy} is increasing for all x, y ∈ X.

Since then many notions and results from classical metric space theory can be extended and generalized to the setting of fuzzy metric spaces. It is proved that every fuzzy metric M on X generates a topology τ_M on X which has as a base the family of open sets of the form B(x, ε, t) = {x ∈ X, 0 < ε < 1, t > 0} where

B(x, ε, t) = {y ∈ X: M(x, y, t) > 1 − ε},

for all ε ∈]0, 1[and t > 0.

The topological space (X, τ) is said to be fuzzy metrizable if there is a fuzzy metric M on X such that $\tau = \tau_M$. Then, it was proved that a topological space is fuzzy metrizable if and only if it is metrizable. From then, several fuzzy notions which are analogous to the corresponding ones in metric spaces have been given, however the theory of fuzzy metric completion is, in this context, very different to the classical theory of metric completion, indeed, there exist fuzzy metric spaces which are not completable. This class of fuzzy metrics can be easily included within fuzzy systems since the value given by them can be directly interpreted as a fuzzy certainty degree of nearness, and in particular, recently, they have been applied to colour image filtering, improving some filters when replacing classical metrics.

Gregori and Sapena[5] defined the concepts of convergent sequene, Cauchy sequene ,completeness and compactness in sense of fuzzy metric space.

They stated that, let M(x, y, t) be a fuzzy metric space, then:

1. A sequence $\{x_n\}$ is said to convergent to x in X, if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all t>0.

2. A sequence $\{x_n\}$ is said to M cauchy, if and only if for each $\varepsilon \in (0,1)$, t > 0, there exist $n_0 \in N$ such that $\lim_{n\to\infty}$ M(x_m, x_n, t)>1-ε for any $m, n \geq n_0$ for all t>0.

3. The fuzzy metric space M(x, y, t) is called M-complete if every M-cauchy sequence is convergent.

4. The fuzzy metric space M(x, y, t) is called compact if every sequence in M has a convergent subsequence.

In 1988 Grabiec[4] introduced the fuzzy version of Banach contraction principle. He fuzzified convergent sequences and Cauchy sequences.

Fang[9] proved some fixed point theorems in fuzzy metric spaces, which improve, generalize some main results of Metric spaces. Sessa[24] defined a generalization of commutativity, which is called weak commutativity in metric space. Pant[17] fuzzify weakly commuting maps as: two mapping f and g of a fuzzy metric space (X, M, $*$) into itself are said to be weakly commuting if, M(fgx, gfx, t) ≥ M(fx, gx, t) for every x∈ X.

Vasuki[25] improved it by defining as R-weakly commuting if, there exist some positive real number R such that, M (fgx, gfx, t) ≥ M (fx, gx, $\frac{t}{R}$) for every x∈ X.Many authors, viz. Gregory and Sapena [14], Mihet D. [14], Saadati and Park [20], Schweizer, Sherwood, Rhoades[19] and Tardiff [24] etc., have proved several fixed point theorems in fuzzy metric space and have given various generalizations of fuzzy metric space.

In 1999 Vasuki [25] obtained the fuzzy version of common fixed point theorem which had extra conditions. In fact, Vasuki[25] proved fuzzy common fixed point theorem by a strong definition of Cauchy sequence.

In 2001 Sapena[21] has given different examples of fuzzy metric spaces.

(i) For each n ∈ N, (X,M, ∧) is a fuzzy metric space where M is given by

M(x,y,t) = $\dfrac{1}{e^{\frac{d(x,y)}{t^n}}}$, for all x,y ∈ X, t > 0, and $\tau_M = \tau(d)$.

(ii) For each k,m ∈ R+, n ≥ 1, (X,M, ∧) is a fuzzy metric space where M is given by

M(x, y, t) = $\dfrac{kt^n}{kt^n + md(x,y)}$ for all x, y ∈ X, t > 0,

and $\tau_M = \tau(d)$ for all x, y ∈ X, t > 0, and $\tau_M = \tau(d)$.

A fuzzy metric (M, ∗) on X is called non-Archimedean if

M(x, z, t) ≥ min{M(x, y, t),M(y, z, t)}, for all x, y, z ∈ X, t > 0.

In 2002 gregori-Romaguera[6] proved that, if (X, M, ∗) is a complete fuzzy metric space then, (X, τ_M) is completely metrizable. They also proved that, a metrizable topological space is compact if and only every compatible fuzzy metric is complete.

Atanassov [2] introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets and later there has been much progress in the study of intuitionistic fuzzy sets.

Park[19] introduced the concept of intutionistic fuzzy sets as a generalization of fuzzy sets as and later there has been much progress in the study of intutionistic fuzzy metric spaces by many authors.

Park[19] stated that an intutionistic fuzzy metric space (IFMS) is a 5-tuple (X, M, N, ∗, ◊) such that X is a non empty set, M and N are fuzzy set on X×X×[0, ∞) satisfying the following conditions, for all x, y, z ∈ X and s, t > 0:

1) M(x, y, t) + N(x, y, t)=1;

2) M(x, y, t) > 0;

3) M(x, y, t) =1 if and only if x=y;

4) M(x, y, t) = M(y,x, t);

5) M(x, y, t) ∗ M(y, z, s) ≤ M(x, z, t+s);

6) M(x, y, .):[0, ∞)→[0, 1] is continuous;

7) N(x, y, t) < 1;

8) N(x, y, t) = 0 if and only if x=y;

9) N(x, y, t) = N(y, x, t);

10) N(x, y, t) ◊ N(y, z, s) ≥ N(x, z, t + s);

In 2009 Gregory, Lopez and Morillas [7] proved that Let (X, M, ∗) and (Y, N, ∗) be two fuzzy metric spaces, D a dense subspace of X and f : D → Y a uniformly continuous mapping. Suppose Y complete. Then, it exists a unique mapping g: X → Y uniformly continuous that extends f . Further, if f is

t-uniformly continuous, then g is t-uniformly continuous.

Gregori and Sapena [8] introduced a class of fuzzy contractive mappings and proved several fixed point theorems in fuzzy metric spaces in the senses of George and Veeramani[3] and Kramosil and Michalek[12], Gregori and Sapena’s [8]results extend classical Banach fixed point theorem and can be considered a fuzzy version of Banach contraction theorem.

In 2004 Mihet [14] defined a new fuzzy contraction called fuzzy ψ -contraction which is a generalization of fuzzy contractive mapping of Gregori and Sapena[8] and proved a fixed point theorem for this kind of contractive mapping in a non-Archimedean fuzzy metric space.

Recently in 2010 Mihet[15] proved Fixed point theorems in fuzzy metric spaces using property (E.A).

Recently in 2013 Shenghua Wang [23] ansered an open question posed by Mihet that whether this fixed point theorem holds if the non-Archimedean fuzzy metric space is replaced by a fuzzy metric space by a theorem that

Let $(X, M, *)$ be an M-complete fuzzy Metric Space in the sense of kramosil and Michalek with $*$ positive, and f:X→X be a fuzzy ψ -contraction mapping. If there is x ∊ X such that $M(x, \ f\,x, t) > 0$, for all t >0, then f has a fixed point in X.

Recently in 2013 M. A. Ahmad and H. A. Nafadi [1] introduced introduce the notion of common limit range property (CLR property) for two hybrid pairs of mappings in fuzzy metric spaces

III. SOME MORE SIGNIFICANT WORK

1. Arzela theorem for fuzzy metric spaces (George and Veeramni, 1995)
2. Continuity and contractivity (Gregori and Sapena, 2001)
3. Fixed point theorems (Gregori, Sapena, 2001)
4. Fixed point theorem in Kramosil and Michalek's fuzzy metric spaces which are complete in Grabiec's sense (Gregori and Sapena, 2001)
5. The construction of the Hausdorff metric on K0(X) (J. Rodríguez-López, S.Romaguera, 2004)
6. Fuzzy quasi-metric spaces (Gregori and Romaguera, 2004) On bicompletion of fuzzy quasi-metric spaces (Gregori, Romaguera and Sapena, 004).
7. The Doitchinov completion of fuzzy quasi-metric spaces (Gregori,.Mascarell and Sapena, 2005)
8. Intuitionistic fuzzy metric spaces (JH. Park, 2004) (Gregori, Romaguera, Veeramani, 2006)
9. Application of fuzzy metrics to color image filtering (Morillas, Gregori, Peris-Fajarnés and Latorre, 2005).

IV. CONCLUSION

It has been observed from various results obtained from the literature survey that there is vast scope of further research to develop new results in fuzzy metric spaces.

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