On Degree of approximation by Product Means of the Fourier Series of a function of Lipchitz class


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Abstract: This paper provides a new approach to a theorem on degree of approximation of Fourier series of a function \( f \in Lip(\alpha, r) \) by \( (E, s)(N, p_n, q_n) \) product summability.

Keywords: Degree of Approximation, \( Lip(\alpha, r) \) class of function, \( (E, q) \) mean, \( (N, p_n, q_n) \) mean, \( (E, s)(N, p_n, q_n) \) product mean, Fourier series, Lebesgue integral.

2010-Mathematics subject classification: 42B05, 42B08.

I. INTRODUCTION:

Let \( \sum a_n \) be a given infinite series with sequence of partial sums \( \{s_n\} \). Let \( \{p_n\} \) and \( \{q_n\} \) be sequences of positive real numbers such that

\[ P_n = \sum_{\nu=0}^{n} p_{\nu} \quad \text{and} \quad Q_n = \sum_{\nu=0}^{n} q_{\nu} \]  

(1)

Let \( t_n = \frac{1}{r_n} \sum_{\nu=0}^{n} P_{n-\nu} q_{\nu} S_{\nu} \)  

(2)

where

\[ r_n = p_0 q_n + p_1 q_{n-1} + \ldots + p_n q_0 (\neq 0) \]

(3)

Then \( \{t_n\} \) is called the sequence of \( (N, p_n, q_n) \) mean of the sequence \( \{s_n\} \). If \( t_n \to s \), as \( n \to \infty \)

then the series \( \sum a_n \) is said to be \( (N, p_n, q_n) \) summable to \( s \).

The necessary and sufficient conditions for regularity of \( (N, p_n, q_n) \) method are[3]:

(i) \( \frac{p_{n-\nu} q_{\nu}}{r_n} \to 0 \) for each integer \( \nu \geq 0 \) as \( n \to \infty \)

and

(4)

(ii) \( \sum_{\nu=0}^{n} |p_{n-\nu} q_{\nu}| < H |r_n| \) where \( H \) is a positive number independent of \( n \).

The sequence –to-sequence transformation [5],

\[ T_n = \frac{1}{(1+q)^n} \sum_{\nu=0}^{n} n q_{n-\nu} s_{\nu} \]  

(6)

defines the sequence \( \{T_n\} \) of the \( (E, q) \) mean of the sequence \( \{s_n\} \). If

\[ T_n \to s \]  

(7)

then the series \( \sum a_n \) is said to be \( (E, q) \) summable to \( s \). Clearly \( (E, q) \) method is regular[5].

Further, the \( (E, q) \) transform of the \( (N, p_n, q_n) \) transform of \( \{s_n\} \) is defined by

\[ \tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} n q_{n-k} t_k \]  

(8)

\[ = \frac{1}{(1+q)^n} \sum_{k=0}^{n} n q_{n-k} \left\{ \frac{1}{r_k} \sum_{\nu=0}^{k} p_{k-\nu} q_{\nu} s_{\nu} \right\} \]  

(9)

If \( \tau_n \to s \), as \( n \to \infty \),

then \( \sum a_n \) is said to be \( (E, q)(N, p_n, q_n) \) summable to \( s \).

Let \( f(t) \) be a periodic function with period \( 2\pi \), \( L \)-integrable over \((-\pi, \pi)\). The Fourier series associated
with \( f \) at any point \( x \) is defined by
\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)
\]
(10)

Let \( s_n(f; x) \) be the \( n \)-th partial sum of (1.10). The \( L_{\infty} \)-norm of a function \( f: R \to R \) by a trigonometric polynomial \( P_n(x) \) of degree \( n \) under norm \( \| \cdot \| \), \( \| \cdot \|_\infty \) is defined by \([11]\)
\[
\| P_n - f \|_\infty = \sup \{ |P_n(x) - f(x)| : x \in R \}
\]
(13)

and the degree of approximation \( E_n(f) \) of a function \( f \in L_\infty \) is given by \([12]\)
\[
E_n(f) = \min_{P_n} \| P_n - f \|_\infty.
\]
(14)

This method of approximation is called Trigonometric Fourier approximation. A function \( f \in \text{Lip} \alpha \) of degree \( \alpha \leq 1 \) \([7]\)
\[
|f(x+t) - f(x)| = O\left(|t|^\alpha\right), \quad 0 < \alpha \leq 1
\]
(15)

and \( f \in \text{Lip}(\alpha, r) \), for \( 0 \leq x \leq 2\pi \), if \([7]\)
\[
\int_0^2 |f(x+t) - f(x)| \, dx = O\left(1\right), \quad 0 < \alpha \leq 1, r \geq 1, t > 0
\]

We use the following notation throughout this paper:
\[
f(t) = f(x+t) + f(x-t) - 2f(x),
\]
(17)

and
\[
K_n(t) = \frac{1}{2\pi (1+s)^k} \sum_{l=0}^{\infty} \left( \frac{n}{k} \right)^s \left( \frac{1}{k} \sum_{r=-q_s}^{q_s} \frac{\sin \left( \frac{r+1}{2} \right)t}{\sin \frac{t}{2}} \right).
\]

Further, the method \( (E, q)\left( N, p_n, q_n \right) \) is assumed to be regular and this case is supposed throughout the paper.

II. KNOWN THEOREMS:

Bernestein\[2\], Alexitis\[1\], Sahney and Goel\[10\], Chandra\[4\] and several others have determined the degree of approximation of the Fourier series of the function \( f \in \text{Lip} \alpha \) by \((C, 1), (C, \delta), (N, p_n)\) and \((\widetilde{N}, p_n)\) means. Subsequently, working on the same direction Sahney and Rao\[11\], and Khan\[6\] have established results on the degree of approximation of the function belonging to the class \( \text{Lip} \alpha \) and \( \text{Lip}(\alpha, r) \) by \((N, p_n)\) and \((\widetilde{N}, p_n, q_n)\) means respectively. However, dealing with product summability Nigam et al \([9]\) proved the following theorem on the degree of approximation by the product \((E, q)\left( C, 1 \right)\) of Fourier series.

Theorem 2.1:
If a function \( f \) is \( 2\pi \)-periodic and of class \( \text{Lip} \alpha \), then its degree of approximation by the product \( \sum_{n=0}^{\infty} A_n(t) \) is given by
\[
\| E_n^q C_n - f \|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1
\]

, where \( E_n^q C_n \) represents the \((E, q)\) transform of \((C, 1)\) transform of \( s_n(f; x) \).

Subsequently Misra et al \([8]\) have established the following theorem on degree of approximation by the product mean \((E, q)\left( N, p_n \right)\) of the Fourier series:

Theorem 2.2:
If \( f \) is a \( 2\pi \)-Periodic function of class \( \text{Lip}(\alpha, r) \), then degree of approximation by the product \((E, q)\left( N, p_n \right)\) summability means on its Fourier series (defined above) is given by
\[
\| \tau_n - f \|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1, r \geq 1
\]

, where \( \tau_n \) as defined in \((8)\).

Extending the above result Paikray et al \([14]\) established a theorem on degree of approximation by the product mean \((E, q)\left( N, p_n \right)\) of the conjugate series of Fourier series of a function of class \( \text{Lip}(\alpha, r) \) . The result is as below:

Theorem 2.3:
If \( f \) is a \( 2\pi \)-Periodic function of class \( \text{Lip}(\alpha, r) \), then degree of approximation by the product \((E, q)\left( \widetilde{N}, p_n \right)\) summability means of the conjugate series \( (10) \) of the Fourier series \((9) \) is given by

ISSN (Print): 2319-2526, Volume -3, Issue -5, 2014
\[ \|r_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha+1/2}}\right), 0 < \alpha < 1, r \geq 1, \]
where \(r_n\) is as defined in (7).

**III. MAIN THEOREM:**

In this paper, we have studied a theorem on degree of approximation by the product mean \((E, s)(N, p_n, q_n)\) of the Fourier series of a function of class \(\text{Lip}(\alpha, l)\). We prove:

Theorem -3.1:

If \(f\) is a \(2\pi\)-Periodic function of the class \(\text{Lip}(\alpha, l)\), then degree of approximation by the product \((E, s)(N, p_n, q_n)\) summability means on its Fourier series (19) is given by

\[ \|r_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha+1/2}}\right), 0 < \alpha < 1, l \geq 1, \]

where \(r_n\) is as defined in (8).

**IV. REQUIRED LEMMAS:**

We require the following Lemma for the proof the theorem.

Lemma -4.1:[14]

\[ |K_n(t)| = O(n), 0 \leq t \leq \frac{1}{n+1}. \]

Lemma-4.2:[14]

\[ |K_n(t)| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \leq t \leq \pi. \]

5. Proof of Theorem 3.1:

Using Riemann –Lebesgue theorem, for the \(n\)-th partial sum \(S_n(f; x)\) of the Fourier series (10) of \(f(x)\) and following Titchmarsh [12], we have

\[ S_n(f; x) - f(x) = \frac{1}{2\pi} \int_0^\pi \phi(t) \sum_{k=0}^{n} \left[ p_n t q_n \frac{\sin\left(\frac{n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \right] dt \]

Using (2), the \((N, p_n, q_n)\) transform of \(S_n(f; x)\) is given by

\[ t_n - f(x) = \frac{1}{2\pi} \int_0^\pi \left\{ \phi(t) \sum_{k=0}^{n} \left[ p_n t q_n \frac{\sin\left(\frac{n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \right] \right\} dt \]

Denoting the \((E, q)(N, p, q)\) transform of \(S_n(f; x)\) by \(r_n\), we have

\[ \|r_n - f\| = \frac{1}{2\pi(1+s)} \int_0^\pi \phi(t) \sum_{k=0}^{n} \left[ p_n t q_n \frac{\sin\left(\frac{n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \right] dt \]

\[ = \frac{1}{\int_0^\pi \phi(t) K_n(t) dt} \]

\[ = I_1 + I_2, \text{ say} \]

\[ |I_1| = \left| \int_0^\pi \phi(t) \sum_{k=0}^{n} \left[ p_n t q_n \frac{\sin\left(\frac{n+1}{2}t\right)}{\sin\left(\frac{t}{2}\right)} \right] dt \right| \]

\[ \leq \int_0^\pi \phi(t) K_n(t) dt \]

\[ = \left( \int_0^\pi \phi(t) \right)^\frac{1}{m} \left( \int_0^\pi K_n(t)^m \right)^\frac{1}{m}, \text{ where } \frac{1}{l} + \frac{1}{m} = 1 \]

, using Holder’s inequality

\[ = O\left(\frac{1}{(n+1)^{\alpha+1/2}}\right) \left( \int_0^\pi n^m dt \right)^\frac{1}{m} \]
\[ = O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \frac{n^m}{n+1} \right) \frac{1}{m} \]
\[ = O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \frac{1}{(n+1)^{\frac{1}{\alpha} + \frac{1}{m}}} \right) \]
Next
\[ |I_2| \leq \left( \int_{n+1}^{\infty} \phi(t) \ dt \right) \left( \int_{n+1}^{\infty} |K_\alpha(t)|^m \ dt \right) \frac{1}{m} \]
using Holder’s inequality, as above.
\[ = O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \int_{n+1}^{\infty} \frac{1}{t} \ dt \right) \frac{1}{m} \]
using Lemma 4.2
\[ = O \left( \frac{1}{(n+1)^{\alpha}} \right) \left( \frac{1}{n+1} \right) \frac{1-m}{m} \]
\[ = O \left( \frac{1}{(n+1)^{\alpha-\frac{1}{\alpha} + \frac{1}{m}}} \right) . \]
Then from (18) and (19), we have
\[ |r_n - f(x)| = O \left( \frac{1}{(n+1)^{\alpha-\frac{1}{\alpha} + \frac{1}{m}}} \right) . 0 < \alpha < l \leq 1. \]
\[ \|r_n - f(x)\| = \sup_{-\pi \leq x \leq \pi} |r_n - f(x)| = O \left( \frac{1}{(n+1)^{\alpha-\frac{1}{\alpha} + \frac{1}{m}}} \right) . 0 < \alpha < l \geq 1. \]
This completes the proof of the theorem.

V. CONCLUSION

The present idea can be extended to establish the theorems on indexed summability factors of Fourier as well as conjugate series. Also other summability methods with order of Lip condition may be taken into consideration.

REFERENCES