An Eco-Epidemic Model with Disease in both Prey and Predator

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Abstract — In the present paper, a four species model is proposed and analyzed. It is considered that the infection floats both in prey and predator. The species once infected cannot be recovered. Consequently, SI model is taken for both the prey and predator populations. It is further assumed that the predator population (both susceptible and infected) consumes only the infected prey. Conditions for the existence and local stability of all the feasible equilibrium points have been carried out.

Keywords: Prey, Predator, Equilibrium points, Stability, Holling Type-II functional Response

I. INTRODUCTION

The study of infectious diseases represents one of the oldest and richest areas in mathematical biology and it is also important and interesting to study ecological systems under the influence of various epidemiological factors. Many researchers had tried to merge ecology and epidemiology and their efforts results in developing a new important branch of mathematical biology named eco-epidemiology [1], [2], [3]. Most of the eco-epidemiological studies are limited to study the situations where only prey species can get infection [4], [5], [6]. A few studies consider the spread of disease in predator species [7, 10, 11].

Many researchers modeled and studied the problem of Salton Sea in California [5, 8, 9]. The Sea is getting polluted by the waste water from Coachella and imperial valleys. The lack of out-flowing streams has resulted in a gradual build up of salts and nutrients in the Sea, causing massive algal blooms in the sea. These algae die as fast as they grow. When it dies, dissolved oxygen from the water is wasted. This process becomes dangerous for the species in the sea especially in summers. Millions of Tilapia dies off every year due to this vivrio infection. This disaster did not stop here. The interaction of the pelican birds (predator of tilapia) with sick tilapia becomes the reason for the infection in birds also. On August 12, 1999, almost 8 millions of the fish died and about 14000 water birds which include mainly white pelicans died after eating the infected tilapia. Many researchers tried and are trying to stop this natural disaster, but they are not able to solve the problem completely.

In the present paper, a predator-prey model is proposed in which both the prey and predator are infected with disease. An SI model is considered both for the predator and prey species as it is assumed that susceptible predator and prey once infected will never recover. Also Holling type-II functional response is considered. Further it is assumed that the predator species (both susceptible and infected) consumes only the infected prey species, as they are low in their mobility and so are easily catachable.

II. MATHEMATICAL MODEL

To form the mathematical model, we take the following assumptions

1. The prey species grows with growth rate r.
2. Both prey and predator populations are invaded by a disease.
3. Mass action incidence rate for transmission of disease.
4. Predator species eats only the infected prey.
5. Holling type-II functional response for predator is considered.

Under above assumptions, following mathematical model is formed.

$$\frac{dx}{dt} = r - \beta xy - d_1 x$$

$$\frac{dy}{dt} = \beta xy - \frac{\alpha_1 yz}{1+y} - d_2 y - \frac{\alpha_2 yw}{1+y}$$

$$\frac{dz}{dt} = z\left(-d_3 + \frac{c_1 y}{1+y}\right) - \lambda_1 zw \qquad (1)$$

$$\frac{dw}{dt} = w\left(-d_4 + \frac{c_2 y}{1+y}\right) + \lambda_1 zw$$

where 'r' is the constant recruitment rate into the prey population, d_1, d_2, d_3 and d_4 are the natural death rates of the susceptible prey, infected prey, susceptible predator and infected predator population respectively, c_1 and c_2 are respective conversion factors of susceptible and infected predator populations where $c_1>c_2$, α_1 and α_2 are their respective search rates. λ_1 is the disease transmission coefficient in predator population.

III. MATHEMATICAL ANALYSIS

The system (1) has following equilibrium points.

(i) $E_1(x_1,0,0,0)$, where $x_1=\dfrac{r}{d_1}$, which always exist.

(ii) $E_2(x_2,y_2,0,0)$, where $x_2=\dfrac{d_2}{\beta}$, $y_2=\dfrac{r\beta-d_1d_2}{d_2\beta}$,

exist if and only if $r\beta-d_1d_2>0$. (2)

(iii) $E_3(x_3,y_3,0,w_3)$

where $x_3=\dfrac{r(c_2-d_4)}{d_1(c_2-d_4)+\beta d_4}$, $y_3=\dfrac{d_4}{c_2-d_4}$,

$$w_3=\frac{c_2}{\alpha_2(c_2-d_4)}\left[\frac{(r\beta-d_1d_2)(c_2-d_4)-d_2d_4\beta}{d_1(c_2-d_4)+\beta d_4}\right]$$

which exist if $c_2>d_4$, $(r\beta-d_1d_2)>\dfrac{d_2d_4\beta}{(c_2-d_4)}$

i.e. $c_2>d_4$, $r>\dfrac{d_2d_4\beta+d_1d_2(c_2-d_4)}{\beta(c_2-d_4)}=\hat{r}_1$(say) (3)

(iv) $E_4(x_4,y_4,z_4,0)$

where $x_4=\dfrac{r(c_1-d_3)}{d_1(c_1-d_3)+\beta d_3}$, $y_4=\dfrac{d_3}{c_1-d_3}$,

$$z_4=\frac{c_1}{\alpha_1(c_1-d_3)}\left[\frac{(r\beta-d_1d_2)(c_1-d_3)-d_2d_3\beta}{d_1(c_1-d_3)+\beta d_3}\right]$$

which exist if $c_1>d_3$, $(r\beta-d_1d_2)>\dfrac{d_2d_3\beta}{(c_1-d_3)}$

i.e. $c_1>d_3$, $r>\dfrac{d_2d_3\beta+d_1d_2(c_1-d_3)}{\beta(c_1-d_3)}=\hat{r}_2$(say) (4)

(v) $E^*(x^*,y^*,z^*,w^*)$

where $x^*=\dfrac{r}{d_1+\beta y}$, $z^*=\dfrac{1}{\lambda_1}\left(d_4-\dfrac{c_2y}{1+y}\right)$,

$$w^*=\frac{1}{\lambda_1}\left(\frac{c_1y}{1+y}-d_3\right)$$

and y^* is the positive root of the equation

$$Ay^{*3}+By^{*2}+Cy^*+D=0 \qquad (5)$$

where A$=-\lambda_1d_2\beta<0$

$$\begin{aligned}B&=\beta\lambda_1r-\beta\alpha_1d_4+\beta\alpha_1c_2-\beta\alpha_2c_1\\&\quad+\beta\alpha_2d_3-\lambda_1d_1d_2-2\beta\lambda_1d_2\end{aligned}$$

$$\begin{aligned}C&=2\beta\lambda_1r-\beta\alpha_1d_4-\alpha_1d_1d_4+d_1\alpha_1c_2+\beta\alpha_2d_3\\&\quad-\alpha_2c_1d_1+\alpha_2d_1d_3-\beta\lambda_1d_2-2\lambda_1d_1d_2\end{aligned}$$

$$D=\beta\lambda_1r-\alpha_1d_1d_4+\alpha_2d_1d_3-\lambda_1d_1d_2$$

Clearly equation (5) has a positive root for y^* if $D>0$

i.e. if $r>\dfrac{d_1d_4\alpha_1-\alpha_2d_3+\lambda_1d_1d_2}{\lambda_1\beta}=\hat{r}_3$(say) (6)

Again z^* and w^* exist if $\dfrac{d_3}{c_1}<\dfrac{y^*}{1+y^*}<\dfrac{d_4}{c_2}$ (7)

The variational matrix for the system is given by

$$\begin{bmatrix}-\beta y-d_1 & -\beta x & 0 & 0\\ \beta y & X & -\dfrac{\alpha_1y}{1+y} & -\dfrac{\alpha_2y}{1+y}\\ 0 & \dfrac{c_1z}{(1+y)^2} & -d_3-\lambda_1w+\dfrac{c_1y}{1+y} & -\lambda_1z\\ 0 & \dfrac{c_2w}{(1+y)^2} & \lambda_1w & -d_4+\lambda_1z+\dfrac{c_2y}{1+y}\end{bmatrix}$$

Where

$$X=\left(\beta x-\frac{\alpha_1z}{1+y}-d_2-\frac{\alpha_2w}{1+y}\right)+y\left(\frac{\alpha_1z}{(1+y)^2}+\frac{\alpha_2w}{(1+y)^2}\right)$$

Following theorems will provide the stability of five equilibrium points of the system (1)

Theorem 1: The equilibrium point $E_1\left(x_1,0,0,0\right)$ is locally asymptotically stable if $\beta r-d_1d_2<0$.

Proof: - The Eigen values at $E_1\left(x_1,0,0,0\right)$ are given by

$$\xi_1=-d_1,\quad \xi_2=\beta x_1-d_2,\quad \xi_3=-d_3,\quad \xi_4=-d_4$$

Clearly $\xi_1,\ \xi_3,\ \xi_4<0$,

and $\xi_2<0$ if $\beta r-d_1d_2<0$ (8)

Theorem 2: The predator free equilibrium point $E_2(x_2,y_2,\ 0,\ 0)$ if exist is locally asymptotically stable if $y_2<\min\left\{\dfrac{d_4}{c_2-d_4},\dfrac{d_3}{c_1-d_3}\right\}$

Proof: - The characteristic equation for the equilibrium point

E_2 are given by

$$\left(\xi+d_4-\frac{c_2y_2}{1+y_2}\right)\left(\xi+d_3-\frac{c_1y_2}{1+y_2}\right)\left(\xi^2+(\beta y_2+d_1)\xi+\beta^2x_2y_2\right)=0$$

Third bracket will have all the eigen values negative as the coefficients are all positive.

The first and second bracket have negative eigen values if

$$y_2<\frac{d_4}{c_2-d_4}\ \text{ and }\ y_2<\frac{d_3}{c_1-d_3}\ .$$

i.e. $y_2<\min\left\{\dfrac{d_4}{c_2-d_4},\dfrac{d_3}{c_1-d_3}\right\}$ (9)

Theorem 3: The equilibrium point $E_3=\left(x_3,y_3,0,w_3\right)$ if

exist is locally asymptotically stable if $\dfrac{y_3}{1+y_3}<\dfrac{\lambda_1w_4+d_3}{c_1}$ and $a_1,a_2>0,\ a_1a_2-a_3>0$, where

$$a_1=\beta y_3+d_1-\alpha_2y_3w_3\Big/\left(1+y_3\right)^2$$

$$a_2=\frac{\alpha_2c_2y_3w_3}{\left(1+y_3\right)^2}-\frac{\alpha_2\beta w_3y_3^2}{\left(1+y_3\right)^2}-\frac{\alpha_2d_1y_3w_3}{\left(1+y_3\right)^2}+\beta\left(\beta x_3-\gamma\right)y_3$$

$$a_3=\frac{\alpha_2c_2\beta y_3{}^2w_3}{\left(1+y_3\right)^2}+\frac{\alpha_2c_2d_1y_3w_3}{\left(1+y_3\right)^2}$$

Proof:-The Characteristic equation corresponding to the equilibrium E_3 are given by the equation

$$\left(\xi+d_3-\frac{c_1y_3}{1+y_3}+\lambda_1w_3\right)\left(\xi^3+a_1\xi^2+a_2\xi+a_3\right)=0$$

, where

$$a_1=\beta y_3+d_1-\alpha_2y_3w_3\Big/\left(1+y_3\right)^2$$

$$a_2=\frac{\alpha_2c_2y_3w_3}{\left(1+y_3\right)^2}-\frac{\alpha_2\beta w_3y_3^2}{\left(1+y_3\right)^2}-\frac{\alpha_2d_1y_3w_3}{\left(1+y_3\right)^2}+\beta\left(\beta x_3-\gamma\right)y_3$$

$$a_3=\frac{\alpha_2c_2\beta y_3{}^2w_3}{\left(1+y_3\right)^2}+\frac{\alpha_2c_2d_1y_3w_3}{\left(1+y_3\right)^2}>0,\ \text{always}$$

Now the first bracket will have negative root if $\dfrac{y_3}{1+y_3}<\dfrac{\lambda_1w_4+d_3}{c_1}$. Again by using Routh- Hurwitz criterion, the second bracket will have negative eigen values if $a_1,a_2>0,\ a_1a_2-a_3>0$.

Theorem 4: The equilibrium point $E_4=\left(x_4,y_4,z_4,0\right)$ if exist is locally asymptotically stable if $\dfrac{y_4}{1+y_4}<\dfrac{d_4-\lambda_1z_4}{c_2}$ and $a_1,a_2>0,\ a_1a_2-a_3>0$, where

$$a_1=d_3-\frac{c_1y_4}{1+y_4}-\frac{\alpha_1y_4z_4}{\left(1+y_4\right)^2}+\left(\beta y_4+d_1\right)$$

$$a_2=\frac{\alpha_1c_1z_4y_4^2}{\left(1+y_4\right)^3}+\frac{\alpha_1c_1z_4y_4}{\left(1+y_4\right)^3}-\frac{\alpha_1d_3z_4y_4}{\left(1+y_4\right)^2}$$

$$+\left(\beta y_4+d_1\right)\left(d_3-\frac{c_1y_4}{1+y_4}-\frac{\alpha_1y_4z_4}{\left(1+y_4\right)^2}\right)+\beta^2x_4y_4$$

$$a_3=\left(\beta y_4+d_1\right)\left(\frac{\alpha_1 c_1 y_4 z_4}{(1+y_4)^3}+\frac{\alpha_1 c_1 {y_4}^2 z_4}{\left(1+y_4\right)^3}-\frac{\alpha_1 d_3 y_4 z_4}{\left(1+y_4\right)^2}\right)$$
$$+\beta^2 d_3 x_4 y_4-\frac{c_1\beta^2 x_4 {y_4}^2}{1+y_4}$$

Proof:-The Characteristic equation corresponding to the equilibrium E_4 are given by the equation

$$\left(\xi+d_4-\frac{c_2 y_4}{1+y_4}-\lambda_1 z_4\right)\left(\xi^3+a_1\xi^2+a_2\xi+a_3\right)=0\ ,$$

where

$$a_1=d_3-\frac{c_1 y_4}{1+y_4}-\frac{\alpha_1 y_4 z_4}{\left(1+y_4\right)^2}+\left(\beta y_4+d_1\right)$$

$$a_2=\frac{\alpha_1 c_1 z_4 y_4^2}{\left(1+y_4\right)^3}+\frac{\alpha_1 c_1 z_4 y_4}{\left(1+y_4\right)^3}-\frac{\alpha_1 d_3 z_4 y_4}{\left(1+y_4\right)^2}$$
$$+\left(\beta y_4+d_1\right)\left(d_3-\frac{c_1 y_4}{1+y_4}-\frac{\alpha_1 y_4 z_4}{\left(1+y_4\right)^2}\right)+\beta^2 x_4 y_4$$

$$a_3=\left(\beta y_4+d_1\right)\left(\frac{\alpha_1 c_1 y_4 z_4}{(1+y_4)^3}+\frac{\alpha_1 c_1 {y_4}^2 z_4}{\left(1+y_4\right)^3}-\frac{\alpha_1 d_3 y_4 z_4}{\left(1+y_4\right)^2}\right)$$
$$+\beta^2 d_3 x_4 y_4-\frac{c_1\beta^2 x_4 {y_4}^2}{1+y_4}$$

Now the first bracket will have negative root if $\frac{y_4}{1+y_4}<\frac{d_4-\lambda_1 z_4}{c_2}$. Again by using Routh- Hurwitz criterion, the second bracket will have negative eigen values if $a_1, a_2, a_3>0,\ a_1 a_2-a_3>0$.

Theorem 5: The interior equilibrium point $E^*\left(x^*,\ y^*,\ z^*, w^*\right)$, is locally asymptotically stable if

$$a_i>0, i=0,1,2,3 \text{ and } a_2 a_3-a_1>0, a_1 a_2 a_3-{a_1}^2-{a_3}^2 a_0>0$$

$$a_0=\left(\beta y^*+d_1\right)\left(\begin{array}{c}\frac{\alpha_2 c_1\lambda_1 y^* z^* w^*}{\left(1+y^*\right)^2}-\frac{\alpha_1 c_2\lambda_1 y^* z^* w^*}{\left(1+y^*\right)^2}\\ -\frac{\alpha_2{\lambda_1}^2 y^* z^* w^{*2}}{\left(1+y^*\right)^2}-\frac{\alpha_1{\lambda_1}^2 y^* z^{*2} w^*}{\left(1+y^*\right)^2}\end{array}\right)$$
$$-\frac{\alpha_1 y^* z^*}{\left(1+y^*\right)^2}-\frac{\alpha_2 y^* w^*}{\left(1+y^*\right)^2}+\beta^2{\lambda_1}^2 x^* y^* z^* w^*$$

where $a_1=(\beta y^*+d_1)\left(\frac{\alpha_2 c_2 y^* w^*}{\left(1+y^*\right)^2}+\frac{\alpha_1 c_1 y^* z^*}{\left(1+y^*\right)^2}+\lambda_1^2 w^* z^*\right)$

$$+\frac{\alpha_2 c_1\lambda_1 y^* z^* w^*}{\left(1+y^*\right)^2}-\frac{\alpha_1 c_2\lambda_1 y^* z^* w^*}{\left(1+y^*\right)^2}-\frac{\alpha_2{\lambda_1}^2 y^* z^* w^{*2}}{\left(1+y^*\right)^2}-\frac{\alpha_1{\lambda_1}^2 y^* z^{*2} w^*}{\left(1+y^*\right)^2}$$

$$a_2=\frac{\alpha_2 c_2 y^* w^*}{\left(1+y^*\right)^2}+{\lambda_1}^2 w^* z^*+\beta^2 x^* y^*\ \frac{\alpha_1 c_1 y^* z^*}{\left(1+y^*\right)^2}$$
$$-(\beta y^*+d_1)\left(\frac{\alpha_1 y^* z^*}{\left(1+y^*\right)^2}+\frac{\alpha_2 y^* w^*}{\left(1+y^*\right)^2}\right)$$

$$a_3=\beta y^*+d_1-\frac{\alpha_1 y^* z^*}{\left(1+y^*\right)^2}-\frac{\alpha_2 y^* w^*}{\left(1+y^*\right)^2}$$

Proof:- The characteristic roots corresponding to the equilibrium E^* are given by the equation

$$\xi^4+a_3\xi^3+a_2\xi^2+a_1\xi+a_4=0$$

By using Routh- Hurwitz criterion, the equation will have negative roots if

$$a_i>0, i=0,1,2,3 \text{ and } a_2 a_3-a_1>0,$$
$$a_1 a_2 a_3-{a_1}^2-{a_3}^2 a_0>0$$

Hence proof.

IV. CONCLUSION

In present paper, a prey predator model is considered in which both the predator and prey species is infected with some disease. An SI model is considered for the both the predator and prey species. Holling type-II functional response is considered. Conditions for the existence and stability of all feasible equilibrium points are obtained. Further it is found that if E_2 exist then E_1 is not locally stable. It is further concluded that E_3 exist if the constant growth rate 'r' is sufficiently large so that the equation (3) is satisfied. Similarly if equation (4) is satisfied, then E_4 exist. Conditions for the existence and stability of non-zero equilibrium point are also obtained.

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