Finite Element Modeling and Free Vibration Analysis of Functionally Graded Nanocomposite Beams Reinforced by Randomly Oriented Carbon Nanotubes

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***Abstract* – This article deals with the finite element modeling and free vibration analysis of functionally graded nanocomposite beams reinforced by randomly oriented straight single-walled carbon nanotubes (SWCNTs). Nanostructural materials can be used to alter mechanical, thermal and electrical properties of polymer-based composite materials, because of their superior properties and perfect atom arrangement. Timoshenko beam theory is used to evaluate dynamic characteristics of the beam. The Eshelby–Mori–Tanaka approach based on an equivalent fiber is used to investigate the material properties of the beam. The equations of motion are derived by using Hamilton's principle. The finite element method is employed to discretize the model and obtain a numerical approximation of the motion equation. Different SWCNTs distributions in the thickness direction are introduced to improve fundamental natural frequency and dynamic behavior of uniform functionally graded nanocomposite beam. Results are presented in tabular and graphical forms to show the effects of various material distributions, carbon nanotube orientations, shear deformation, slenderness ratios and boundary conditions on the dynamic behavior of the beam. The first five normalized mode shapes for functionally graded carbon nanotube reinforced composite (FG-CNTRC) beams with different boundary conditions and different carbon nanotubes (CNTs) orientation are presented. The results show that the above mentioned effects play very important role on the dynamic behavior of the beam.**

Keywords – Carbon nanotubes, Mori–Tanaka approach, FG beam, Finite element analysis, Vibration.

I. INTRODUCTION

Functionally Graded Material (FGM) belongs to a class of advanced material characterized by variation in properties as the dimension varies. The overall properties of FMG are unique and different from any of the individual material that forms it.FGM concept was originated in Japan in 1984 during the space plane project.

Carbon nanotube (CNT) is a new form of carbon, configurationally equivalent to two dimensional graphene sheet rolled into a tube. It is grown now by several techniques in the laboratory and is just a few nanometers in diameter and several microns long. CNT exhibits extraordinary mechanical properties: the Young's modulus is over 1 Tera Pascal. It is stiff as diamond. The estimated tensile strength is 200 Giga Pascal. These properties are ideal for reinforced composites, nanoelectromechanical systems (NEMS).Carbon Nanotubes (CNTs) are also known for their good mechanical and electrical properties. The use of CNT with Functionally Graded Material (FGM) provides improved mechanical, electrical as well as thermal properties. Main advantage of using CNT based FGM is that we can obtain these properties as per our requirement just by varying the distribution and composition of CNT. The CNT based functionally graded materials are used in wind turbines, tissue engineering, thin films of shape memory alloys, nano-electromechanical systems micro sensors, micro actuators, telecommunications and transport industry.

Most studies on carbon nanotube-reinforced composites (CNTRCs) have focused on their material properties [2–7]. Several investigations have shown that the addition of small amounts of carbon nanotube can considerably improve the mechanical, electrical and thermal properties of polymeric composites [4–7]. Wuite and Adali [5] examined the deflection and stress of nanocomposite reinforced beams using a multi-scale analysis. They found that a small percentage of nanotube reinforcement leads to significant improvements in beam stiffness. Studies showed that the addition of a small amount of carbon nanotube can considerably improve the mechanical, electrical and thermal properties of polymeric composites. Their results are very useful and can be applied to the analysis of the global response of CNTRC in an actual structural element. Liao-Liang Ke et al. [10] investigated the nonlinear free vibration of functionally graded nanocomposite beams reinforced by aligned, straight single-walled carbon nanotubes (SWCNTs) based on Timoshenko beam theory. The material properties of functionally graded carbon nanotube-reinforced composites (FG-CNTRCs) were assumed to be graded in the thickness direction and estimated though the rule of mixture. They introduced the CNT efficiency parameter to account for load transfer between the nanotube and polymeric phases. However, the rule of mixture is not applicable when CNTs are oriented randomly in the matrix. Thus, in this paper the Mori–Tanaka approach which is applicable to nanoparticle is employed to predict material properties of composites reinforced with randomly oriented, straight CNTs.

The objective of the present work is to study the free vibrations of functionally graded nanocomposite beams reinforced by randomly oriented straight single-walled carbon nanotubes within the framework of Timoshenko beam theory using finite element method. The material properties of the FG-CNTRC are assumed to be graded in the thickness direction and estimated though the Mori–Tanaka method **[11]** because of its simplicity and accuracy even at a high volume fraction of inclusions. Finally, the effects of CNTs orientation, Effect of variation of volume fraction of CNT, slenderness ratios and boundary conditions on the dynamic characteristics of the beam are investigated. Using finite element analysis eigenvalues are evaluated for different boundary conditions. Mode shapes are plotted to visualize vibration response.

II. MATERIAL PROPERTIES OF FG-CNTRC

2.1 Properties of equivalent fiber:

Using the results obtained from multi-scale FEM, the investigated CNT and its inter-phase can be converted into an equivalent fiber. Thus an embedded carbon nanotube in a polymer matrix is replaced with an equivalent long fiber for predicting the mechanical properties of the carbon nanotube/polymer composite. The equivalent fiber for SWCNT with chiral index of (10,10) is a solid cylinder with diameter of 1.424 nm. The rule of mixture is used inversely for calculating material properties of equivalent fiber [13]:

$$E_{LEF} = \frac{E_{LC}}{V_{EF}} - \frac{E_M V_M}{V_{EF}}$$

$$\frac{1}{E_{TEF}} = \frac{1}{E_{TC} V_{EF}} - \frac{V_M}{E_M V_{EF}}$$

$$\frac{1}{G_{EF}} = \frac{1}{G_C V_{EF}} - \frac{V_M}{G_M V_{EF}}$$

$$\upsilon_{EF} = \frac{\upsilon_C}{V_{EF}} - \frac{\upsilon_M V_M}{V_{EF}}$$

where E_{LEF}, E_{TEF}, G_{EF}, υ_{EF}, E_{LC}, E_{TC}, G_C, υ_C, E_M, G_M, υ_M, V_{EF} and V_M are longitudinal modulus of equivalent fiber, transverse modulus of equivalent fiber, shear modulus of equivalent fiber, Poisson's ratio of equivalent fiber, longitudinal modulus of composites, transverse modulus of composites, shear modulus of composites, Poisson's ratio of composites, modulus of matrix, shear modulus of matrix, Poisson's ratio of matrix, volume fraction of the equivalent fiber and volume fraction of the matrix.

Table 1: Material properties of equivalent fiber:

Mechanical property	Equivalent fiber [9]
Longitudinal Young's modulus (E_{LEF})	649.12 (GPa)
Transverse Young's modulus (E_{TEF})	11.27 (GPa)
Longitudinal shear modulus (G_{EF})	5.13 (GPa)
Poisson's ratio (υ_{EF})	0.284

The modeling of beam starts with the calculation of material properties. As CNT volume is assumed to vary along the thickness only, material properties for each layer are calculated first and finally effective values for entire beam are calculated. Here, linear variation of volume fraction of CNT (V_{cnt}) is considered. It is calculated by following formula:

$$V_{cnt} = \frac{4|z|}{h} V_{cnt}^*$$

Here, V_{cnt}^* depends on mass fraction and density of CNT and density of matrix. For unidirectional CNT distribution $V_{cnt} = V_{cnt}^*$.

2.2. Composites reinforced with randomly oriented, straight CNTs:

The effect of randomly oriented, straight CNTs is investigated in this section. The orientation of a straight CNT is characterized by two Euler angles α and β, as shown in Fig. 1. The base vectors e_i and e'_i of the global (0 -$x_1x_2x_3$) and the local coordinate systems (0- $x'_1x'_2x'_3$) are related via the transformation matrix g:

$$e_i = g_{ij} e'_j$$

Where g is given by:

$$g = \begin{bmatrix} \cos\beta & -\cos\alpha\sin\beta & \sin\alpha\sin\beta \\ \sin\beta & \cos\alpha\cos\beta & -\sin\alpha\cos\beta \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

The orientation distribution of CNTs in a composite is characterized by a probability density function p(α,β) satisfying the normalizing condition [3].

$$\int_0^{2\pi}\int_0^{\pi/2} p(\alpha,\beta)\sin\alpha \, d\alpha \, d\beta = 1$$

If CNTs are completely randomly oriented, the density function is:

$$p(\alpha,\beta) = \frac{1}{2\pi}$$

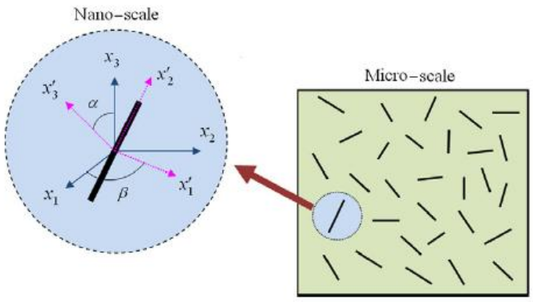

Fig.1.Representative volume element (RVE) with randomly oriented, straight CNTs. [12]

We found the Hill's elastic moduli of the reinforcing phase from the equality of two following matrices [13]:

$$C_r = \begin{bmatrix} n_r & l_r & l_r & 0 & 0 & 0 \\ l_r & k_r+m_r & k_r-m_r & 0 & 0 & 0 \\ l_r & k_r-m_r & k_r+m_r & 0 & 0 & 0 \\ 0 & 0 & 0 & p_r & 0 & 0 \\ 0 & 0 & 0 & 0 & m_r & 0 \\ 0 & 0 & 0 & 0 & 0 & p_r \end{bmatrix}$$

$$C_r = \begin{bmatrix} 1/E_{LEF} & -\upsilon_{31}/E_{TEF} & -\upsilon_{31}/E_{TEF} & 0 & 0 & 0 \\ -\upsilon_{EF}/E_{LEF} & 1/E_{TEF} & -\upsilon_{31}/E_{TEF} & 0 & 0 & 0 \\ -\upsilon_{31}/E_{LEF} & -\upsilon_{31}/E_{TEF} & 1/E_{TEF} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{EF} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{EF} \end{bmatrix}^{-1}$$

Where Cr is the stiffness tensor of the equivalent fibers and

$$\upsilon_{31} = \frac{E_{TEF}\upsilon_{EF}}{E_{LEF}}$$

$$G_{23} = \frac{E_{TEF}}{2(1+\upsilon_{EF})}$$

When CNTs are completely randomly oriented in the matrix, the composite is then isotropic, and its bulk modulus K and shear modulus G are derived as **[3]**

$$K = K_m + \frac{V_{cnt}(\delta r - 3K_m\alpha_r)}{3(V_m + V_{cnt}\alpha_r)}$$

$$G = G_m + \frac{V_{cnt}(\eta_r - 2G_m\beta_r)}{2(V_m + V_{cnt}\beta_r)}$$

$$\alpha_r = \frac{3(K_m+G_m)+k_r-l_r}{3(G_m+k_r)}$$

$$\beta_r = \frac{1}{5}\left\{ \frac{4G_m+2k_r+l_r}{3(G_m+k_r)} + \frac{4G_m}{G_m+p_r} + \frac{2\left[G_m(3K_m+G_m)+G_m(3K_m+7G_m)\right]}{G_m(3K_m+G_m)+m_r(3K_m+7G_m)} \right\}$$

$$\delta_r = \frac{1}{3}\left[n_r + 2l_r + \frac{(2k_r+l_r)(3k_m+2G_m-l_r)}{G_m+k_r} \right]$$

$$\eta_r = \frac{1}{5}\left[\frac{2}{3}(n_r-l_r) + \frac{8G_m p_r}{G_m+p_r} + \frac{8m_r G_m(3K_m+4G_m)}{3Km(m_r+G_m)+G_m(7m_r+G_m)} + \frac{2(k_r-l_r)(2G_m+l_r)}{3(G_m+k_r)} \right]$$

Where k_r, l_r, m_r, n_r, and p_r are the Hill's elastic moduli for the reinforcing phase (CNTs).

III. PROBLEM FORMULATION

Based on the first-order shear deformation (or the Timoshenko beam) theory, the axial displacement U and the transverse displacement of any point of the beam, W, are given by [1]

$$u(x,y,z,t) = u_0(x,t) - z\phi(x,t)$$

$$w(x,y,z,t) = w_0(x,t)$$

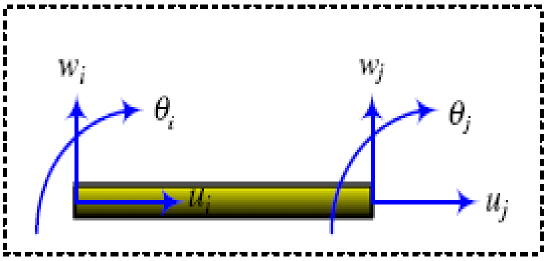

Fig.2 : Beam Element [14]

Strain–displacement and constitutive relations are formed to calculate strain energy and kinetic energy of beam. For given case the final expressions for these energies are,

$$U_b = \frac{1}{2}\int_v (\sigma_{xx}\varepsilon_{xx} + \tau_{xz}\gamma_{xz})dAdx$$

$$T_b = \frac{1}{2}\int_v (\dot{u}^2 + \dot{w}^2)dAdx$$

Hamilton's principle is applied to get the differential equations in terms of two translational and one rotational degree of freedom i.e. u, w and ϕ [13].

$$\delta u: I_0(x)\ddot{u} - I_1(x)\ddot{\phi} - \frac{\partial}{\partial x}\left(A_{11}(x)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(B_{11}(x)\frac{\partial \phi}{\partial x}\right) = 0$$

$$\delta w: I_0(x)\ddot{w} - \frac{\partial}{\partial x}\left(A_{55}(x)\frac{\partial w}{\partial x} - \phi\right) = F_0\delta(x - vt)$$

$$\delta\phi: I_2(x)\ddot{\phi} - I_1(x)\ddot{u} + \frac{\partial}{\partial x}\left(B_{11}(x)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial x}\left(D_{11}(x)\frac{\partial \phi}{\partial x}\right) - k_s A55(x)\left(\frac{\partial w}{\partial x} - \phi\right) = 0$$

3.1. Finite Element Analysis

To solve the above governing equation finite element analysis is implemented. Finite element analysis aim is to find out the field variable (displacement) at nodal points by approximate analysis. Here employing the approximate solution method the governing equations are approximated by a system of ordinary differential equations. Finally, the equation of motion is derived as follows:

$$[M]\{\ddot{q}\} + [K]\{q\} = \{F\}$$

Where $[M]$ denotes the mass matrix, $[K]$ denotes the stiffness matrix. The elements of stiffness and mass matrix are as given by [13].

IV. RESULTS AND DISSCUSSION

This theory has been implemented to a beam with *h* = 1 m, *L/h* = 20 to study the convergence of dimensionless fundamental frequency for V_{cnt}^* = 0.075.It is calculated by following formula:

$$\lambda^2 = \omega L^2 \sqrt{\frac{\rho_m A}{E_m I}}$$

Now, with the material properties E_m= 10GPa; ν_m = 0.3 and ρ_m = 1150 Kg/m^3 results of the first five non-dimensional frequencies of clamped–clamped (C–C), SFG-CNTR beams based on Timoshenko beam theory with different number of element are obtained and is shown in table 2. It is observed that the convergence of the present results is occurs with a number of element *N* = 100. To validate the results beam slenderness ratio *L/h* = 20 and V_{cnt}^* = 0.075 are selected with clamped–clamped condition of Timoshenko beam and are verified with the results given by **[13]** .The effect of variation of elements on dimensionless fundamental frequency is shown in table 2.

Table 2.Effect of number of elements on non-dimensional fundamental frequency:

Mode No.	20	40	60	80	100
1	5.7046	5.4562	5.4229	5.4089	5.4030
2	7.7632	7.7280	7.6686	7.6506	7.6413
3	9.6638	9.4572	9.3968	9.3708	9.3599
4	9.8956	10.9441	10.8519	10.8237	10.8090
5	11.7088	12.2242	12.1418	12.1037	12.0874

Table 3 shows effect of slenderness ratio (*L/h*) on dimensionless fundamental frequency. Frequencies are calculated for slenderness ratio = 20, 40, 60, 80. It is found that as *L/h* ratio increases frequency also increases.

Table 3. Effect of slenderness ratio on non-dimensional fundamental frequency:

Mode No.	L/h = 20	L/h = 40	L/h = 60	L/h = 80
1	5.4030	7.6883	9.5166	11.1623
2	7.6413	10.8722	13.4554	15.7780
3	9.3599	13.3185	16.4855	19.3357
4	10.8090	15.3790	19.0325	22.3165
5	12.0874	17.1994	21.2886	24.9678

Table 4 compares the effect of variation of volume fraction of CNT on dimensionless fundamental frequency. The first five frequencies for Clamped–Clamped, SFG-CNTRC beams are shown in the table:

Table 4. Effect of variation of volume fraction of CNT on dimensionless fundamental frequency

Mode No.	V_{cnt}				
	0.075	0.1	0.125	0.15	0.175
1	5.4030	5.7963	6.1273	6.4122	6.6630
2	7.6413	8.1981	8.6656	9.0706	9.4244
3	9.3599	10.0411	10.6146	11.1080	11.5424
4	10.8090	11.5967	12.2579	12.8307	13.3311
5	12.0874	12.9672	13.7077	14.3448	14.9053

Corresponding Mode Shapes are plotted for first 5 natural frequencies for first case of table 4. Figure 3-7 shows mode shape for first case with V_{cnt} =0.075.

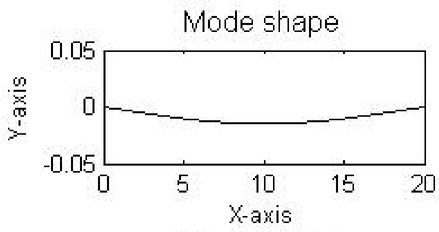

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Fig. 3 : Mode shape for first mode

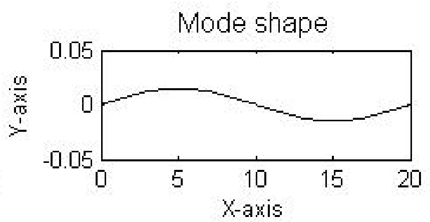

Fig. 4 : Mode shape for second mode

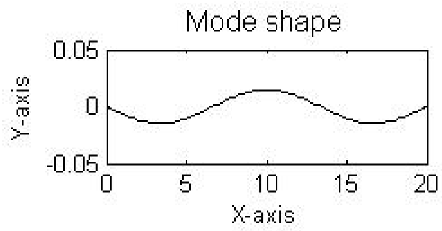

Fig. 5 : Mode shape for third mode

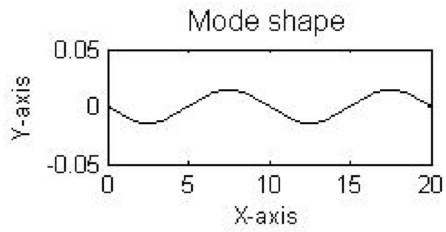

Fig. 6 : Mode shape for fourth mode

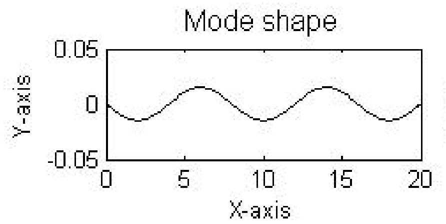

Fig. 7 : Mode shape for fifth mode

Displacement versus time diagram are plotted to understand the time response with varying CNT volume fraction. Fig.8. shows the response of:

i. Sinusoidal force of magnitude 100 N which is acting at the center of the beam.

ii. With clamped-clamped boundary condition of FG beam and exciting frequency of 80 Hz.

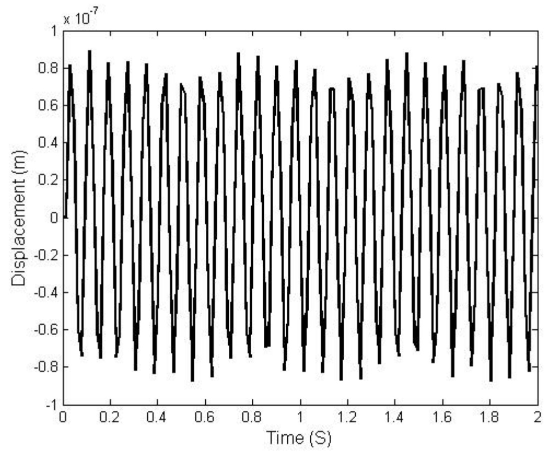

Fig. 8 : Plot of Displacement Vs time of CNT based FG beam under sinusoidal loading

V. CONCLUSION

CNT based FG Timoshenko beam has been modeled using finite element method. Firstly convergence of results has been studied and then free vibrations as well as dynamic analyses have been carried out. Nondimensional fundamental frequencies are calculated for different volume fractions of CNT and slenderness ratio. Mode shapes and displacement history are also presented for this beam.

Results show that as volume fraction of CNT increases, fundamental frequency also increases. The reason behind this is more volume of CNT provides more stiffness to the beam which results in higher frequencies. Similar effect is obtained for frequencies with increase in the slenderness ratio. Non dimensional fundamental frequency increases as slenderness ratio increases. The response of displacement versus time shows that for increasing values of volume fraction displacement goes on reducing.

Finally it can be concluded that, as volume fraction increases, stiffness of the beam increases and it results in increased value of fundamental frequency and reduction in deflection.

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